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MEMOIR

ON

THE THEORY

OF PARTIAL FUNCTIONS,

BY ITS AUTHOR,

JOHN WALSH.

The Lemmas of Euclid and Newton are Mathematically False,
GEOMETRIC BASE.

CORK:

PRINTED BY JAMESON AND MARTIN.

1836.

INTRODUCTION.

THE following Memoir has been drawn up in explanation of my Theory, and in reply to the objections to this singularly elegant and powerful Geometry, by the Royal Society of London, and the National Institute of France, expressed in these words, which I transcribe from their notes to me on that subject.

June 1st, 1832.

THE Committee have decided that your Memoir is unworthy the attention of the Royal Society.

Signed, ROGET, SEC. to the ROYAL SOCIETY.

June 16th, 1823.

YOUR Commissioners advise the Academy no longer to waste its time on the Memoirs of M. Walsh.

Signed, POISSON, CAUCHY, REPORTER.

To these notes, which clearly persuade me, that the Geometricians of Great Britain and France are not only very bad scholars, but very prejudiced men, and as the present Memoir will fully prove, I now return the following answer.

SECTION 1.

ON THE THEORY OF TANGENTS.

Let the curve OPZ *fig* 1. be the common parabola, and a its parameter, that is a given right line. Let O I and I P be the rectangular coordinates of any given point P; and let O I′ and I′ P′ be the coordinates of any other point P′; both pair of coordinates I refer to the origin O and the axis X; but with regard to the origin P and the axis X′ passing through P and parallel to the former, the rectangular coordinates of the point P′ are P N′ and N′ P′; so that while the point P, has only one pair of coordinates, the point P′ has two.

Let now p be put for O I, and q for I P, then from the nature of this curve, the relation between O I and I P, is expressed by the equation,

$$q = \sqrt{ap}$$

putting, now, x' for O I′ and y' for I′ P′, then we shall have similarly to the former, the relation y' equal to $\sqrt{ax'}$, or by putting $p + x$ in place of x', the general equation then will be $y' = \sqrt{q^2 + ax}$, and putting x for P N′, and y for N′ P′, then the relation between the coordinates P N′ and N′ P′, will be expressed by the equation.

$y = -q + \sqrt{ax'}$, or, $y = -q + \sqrt{q^2+ax}$. So that we have altogether three pair of coordinates, that is to say, one for the origin, O, and axis X, one for the same origin O, point P′ and the same axis X; and one for the same point P′, and origin P, and the axis X′.

I will now explain the nature of the terms which I employ in this Memoir, and which, if well recollected, will leave little difficulty in understanding my theory to any person conversant in the principles of algebra.

The axis X, I call the local axis, and O the local origin; I call X′ the partial axis, and P the partial origin. I call P also the local point, and P′ the general point. With regard to P and origin O, I call the coordinates local, and with regard to P′ and the same origin O, I call them *general; and with regard*

to the point P', axis X', and the origin P, I call the coordinates P N' and N' P', the partial coordinates of the curve. This simple invention of partial coordinates, which took place in number 12, Tuckey Street, Cork, at eleven of the clock, June 24, 1822, has accomplished a full, complete and total revolution in the higher mathematics, and it marks unquestionably the most remarkable era in the history of human knowledge.

With regard to P and origin O, I call the equation of the curve local; and with regard to P' and origin O, I call the equation *general*; and with regard to P' and the origin P, I call the equation of the curve its *partial equation.* The arc O P is the local arc, O P' is the general arc, and P P' is the partial arc. O I P is the local area, O I' P' is the general area, and I I' P' P is the partial area, I call p and q, indeterminate constants, and I call, x an independent variable, &c. I employ similar terms with regard to curve surfaces and solids, &c. So that in general I make use of the terms, *local, general,* and, *partial functions;* and as either of these may be greater than, the same as, or less than, either of the other two, I call the application of them *the theory of partial functions.* It is the only true theory, the deepest, the most subtle, and the most general theory that has been yet invented; it embraces within its controul, all the practical rules before known so far as they were exact, and far, very far transcends them in the extent and variety of its applications, as I will now proceed to demonstrate. And first of all that there should be no loop hole for error to creep in, I will lay down my three rules of reasoning, my demonstration of the law of evolution, and my definition of angular space, which definition is the only one that explains the real nature of that species of magnitude, as it is employed in geometry, and without which it is impossible to demonstrate the theory of parallel lines.

1. No proposition containing more than two ideas is admitted without demonstration.

2. No supposition is admitted that cannot be demonstrated to be possible.

3. No demonstration is founded on the truth of the proposition to be demonstrated.

With regard to evolution, if we had to elevate $p+x$, to the power 2, the common rule to find the coefficient of each power of x, from that which next precedes it, is to multiply by the exponent, then diminish the exponent by unity, and divide by the number of terms up to that place, in which the first term is $p^2 x^0$. The coefficient of x^0, is p^2, which, when multiplied by 2, and the exponent diminished by unity becomes $2p^{2-1}$, and the second term then will be $2p^{2-1}x$, similarly the third term will be $\frac{2(2-1)}{1 \cdot 2} p^{2-2}x^2$, and so on. So that we have then by a rule which was known to Cardan and to the Arabians before him,

$$(p+x)^2 = p^2 + 2p^{2-1}x + \frac{2(2-1)}{1 \cdot 2} p^{2-2}x^2 + \frac{2(2-1(2-2)}{1 \cdot 2 \cdot 3} p^{2-3}x^3 + \&c.$$

And now I say, that if for 2, I substitute n, then for all mathematical values of n, I shall have similar in form to the above,

$$(p+x)^n = p^n + np^{n-1}x + n\frac{(n-1)}{2} p^{n-2}x^2 + \&c.$$

and putting z for the whole of the series after the first term, or

$$z = np^{n-1}x + \frac{n(n-1)}{2} p^{n-2}x^2 + \&c,$$

that I shall have similarly,

$$\left(p^n+z\right)^{\frac{1}{n}} = p+x = p + \tfrac{1}{n}\left(p^n\right)^{\frac{1}{n}-1}z + \&c. \text{ or}$$

$$x = \tfrac{1}{n}\left(p^n\right)^{\frac{1}{n}-1} z + \tfrac{1}{n}\frac{(\frac{1}{n}-1)}{2}\left(p^n\right)^{\frac{1}{n}-2}z^2 + \&c$$

For in the series for z, substituting for x, the series for x, there results z equal to z; or in the series for x, substituting for z, the series for z, there results x equal to x. Wherefore the law of evolution is the same for all mathematical values of the exponent.

In regard to a rectilineal angle, it is the plane surface, bounded only by two right lines traced without bounds from the same point

Because space is infinite, it results from this definition, that in the measurement of angular space, all parallel infinite planes are to be taken as all coinciding with one another—and all parallel infinite right lines as coinciding with one another—and all bounded space a point; for were it otherwise, then space would not be infinite, which is absurd.

From this it results, that as the area of a triangle is zero in the measurement of the areas of its angles, then the sum of these areas is equal to the area of two right angles.

Therefore, the theory of parallels, the basis of geometry, is demonstrated.

The analogy between plane triangles, and spherical triangles, is remarkable in this instance.

The area of the three angles of every plane triangle, is equal to the area of two right angles, together with twice the area of the triangle; and in spherical triangles, the area of the three angles, is equal to the area of two right angles, together with twice the area of the triangle.

A spherical angle, is either of the divisions of the surface of a sphere made by two great circles, in the same manner as a plane angle, is either of the divisions of an infinite plane made by two infinite straight lines.

Having demonstrated the theory of parallels, the basis of plane geometry, and the law of evolution, the basis of the geometry of curves, &c. I will now proceed to the solution of the question of tangents; and, before all, the reader should make himself perfectly familiar with the explanations I have given above.

Taking the general equation of the parabola, $\sqrt{q^2+ax}$, it becomes, by evolution, that is, the general ordinate becomes,

$$I'P' = q + \frac{q}{2p}x - \frac{q}{8p^2}x^2 + \&c.$$

Now, as x, is the only variable here, because p and q are constant, it is evident that the first two terms, or $q + \frac{q}{2p}x$,

form the general equation of a right line, passing through the local P, which is the origin of x, that is, x is measured on the partial axis, X′, to the right and left from P, and $q + - \frac{q}{ap} x$ is the general ordinate of a right line passing through P. And to explain myself more clearly, on any general ordinate I′ P′ produced if necessary. I take N′ T′ equal to $\frac{q}{2p} x$, that is, so that, N′ T′ will be a fourth proportional to $2p$, q, and x. Then because I′ N′ is equal to q, it follows that I′ T′ is equal to $q + \frac{q}{2p} x$; and T′ T passing through P, and meeting the axis X in T, is the line of which $q + \frac{q}{2p} x$ is the general ordinate.

Now it is remarkable, that this same line T′ P T is a tangent at P. And to prove this, it is only necessary to show, that I′ T′ is greater than I′ P′. Now I′ P′ is equal to $\sqrt{q^2 + ax}$; and I′ T′ is equal to $q + \frac{q}{2p} x$. The square of the former is, $q^2 + ax$, and the square of the latter is $q^2 + ax + (\frac{q}{2p} x)^2$. Therefore the square of I′ T′ is equal to the sum of the squares of I′ P′ and N′ T′, and therefore T′ P T falls outside the curve, and is a tangent at P, since I′ T′ is greater than I′ P′.

The ordinate of the tangent to the left of P, is $q - \frac{q}{2p} x$, and as at T where the tangent intersects the local axis, this ordinate is nothing; then, in that case, $q - \frac{q}{2p} x$, is equal to nothing, and q is equal to $\frac{q}{2p} x$, and then x is equal to $2p$; that is the substangent I T, to which the partial abscissa x, taken to the left of P, is then equal, is itself equal to twice O I.

Let now P C be drawn perpendicular to the tangent, then as the ordinate of the tangent when refered to P, as origin

is $\frac{q}{2p}x$, which is equal to, N′ T′; so the partial ordinate of the normal P C, is N′ C′, and to determine it, we have, N′ T′ : N′ P :: N′ P : N′ C′; or which is the same, $\frac{q}{2p}x : x :: x : \frac{2p}{q}x$. So that N′ C′ is equal to $\frac{2p}{q}x$, which is the partial equation of the normal P C. And since I′ N′ is equal to q, then is I′ C′ equal to, $-q+\frac{2p}{q}x$, which is the general equation of the normal, the origin being at P. Now, as at the point C where the normal meets the local axis, the ordinate I′ C′ is zero, then in that case, we have, $-q+\frac{2p}{q}x$, equal to zero; and then x is equal to, $\frac{q^2}{2p}$, or to $\frac{ap}{2p}$, or to $\frac{a}{2}$; that is to say, the subnormal I C, to which the partial abscissa, x, taken to the right of P, is then equal, is itself equal to half the parameter, a. And as the normal is the shortest line that can be drawn from C to the curve; and as it is perpendicular to the tangent, it may be proved that the parabola cannot have two tangents at the same point, which I will presently show.

Let us now examine if the partial axis intersects this curve at any point but P. In case it did, then at that point the local and general ordinates would be equal, since the two axes X and X′ are parallel, and we would have in that case,

$$q=\sqrt{q^2+ax}, \text{ or } q^2=q^2+ax, \ldots m,$$

the less equal to the greater, which is absurd; and therefore the partial axis does not intersect this curve but in one point; and for a similar reason, the partial axis cannot be a tangent to the parabola at any point. And since it has been shown that the general ordinates of the parabola and tangent are always unequal, its tangent can no where intersect the parabola.

I will now take the first three terms of the general series; these are $q+\frac{q}{2p}x-\frac{q}{8p^2}x^2$; and they form the equation

of a parabola, n P n', secant to the given parabola at P. Its relation to the given curve is determined in the same manner as that of the tangent and partial axis. In effect, to determine the point where it intersects the local axis, we have the equation,

$$q+\frac{q}{2p}x-\frac{q}{8p^2}x^2=o.$$

and for the point, n', where it intersects the partial axis we have,

$$\frac{q}{2p}x-\frac{q}{8p^2}x^2=o, \text{ or } x=\mathrm{P}n'=2\,\mathrm{P}g$$

and where it intersects the curve is determined by the equation, $q+\frac{q}{2p}x-\frac{q}{8p^2}x^2=\sqrt{q^2+ax}$, or $x=o$, or, $x=8p$. The line of the local axis of this curve, is $h\ gg'$; and the line of its partial axis is P I. Its vertex is h and gh, is half $g\,m'$ which itself is equal to q. If we refer the equation of this secondary parabola to its local axis, it will be, $q'^2=\frac{8p^2}{q}p'$; having $\frac{8p^2}{q}$ for parameter, its subtangent gm' is equal to q, and its subnormal gg' is equal to $\frac{4p^2}{q}$; and it has the same tangent as the given parabola, and the same radius of curvature, or what is called such.

Taking the four terms, $q+\frac{q}{2p}x-\frac{q}{8p^2}x^2+\frac{q}{16p^3}x^3$, they form the equation of a parabola, m P m', tangent to the given curve, which it no where intersects; the point where it intersects the local axis, is determined by making the whole equation equal to zero, being a cubic equation, having only one real root. The point m', where this parabola intersects the tangent straight line, is determined by making the equations of both tangents equal to each other, which gives x equal to Pg, equal to twice p; whence it is, that the local axis of the secant parabola, the tangent straight line, and the tangent parabola, intersect each other in the same point. If we take five terms of the series, then the secondary parabola v P v' is a secant to the given parabola; and so on alternately, secant and tangent. The primary and secondary curves, have the same straight line tangent T T'; but it may be observed, that while T T' is a

tangent to the curve m P m', at P it intersects it at m'. Now this point, which marks the limit of the theorem of inequality, with regard to the tangent to this curve, considered as a primary one, is determined, by making the equation of the tangent equal to the undeveloped function, and the rule is the same for every other secondary, as I have shown above; and after these limits, the theorem of inequality is reversed. And to explain myself more clearly, if we compare the four terms which compose the general equation of this tangent curve, to the first two terms which compose the equation of its tangent straight line, we shall find that from P to m', the first two are together greater than all four; that is, N, T, the ordinate of the tangent is greater than N' i, the partial ordinate of the curve m P m'; and for this reason, that $\frac{x^2}{8p^2}$ is greater than $\frac{x^3}{16p^3}$ from P to m', or so long as x is not greater than P g, or than its equal $2p$ equal twice O I. Let $-\frac{x^2}{8p^2}+\frac{x^3}{16p^3}$ be equal to $-k$ within this limit, and then evidently the ordinate $q+\frac{q}{2p}x$ of the tangent is greater than the general ordinate, $q+\frac{q}{2p}x-k$ of the curve. But after the limit determined by the equation, $-k$ equal zero, although the third term is essentially inverse, the sum of the first two terms is less than all four. In the same manner, because the sign of the fourth term is essentially direct, the sum of the first three terms is less than all four, so long as five times x is not greater than eight times p; and so on. Such are the limits by which the intersections of both secondaries and primaries with one another, are determined; and such are the limits of the theorem of inequality; but in all general series, so long as the partial abscissa is not greater than the local abscissa, the theorem of inequality is unlimited; it is demonstrated at page twelve of my work, the Geometric Base.

Section II.

ON MAXIMA AND MINIMA.

As the problem of tangents is imperfect without the aid of maxima and minima, I will now proceed to demonstrate the general principles of the latter.

Let it be proposed to draw from a given point C in the local axis of the parabola, the shortest line C P to the curve.

As the point O is given, I put c for O C, and then will I C be equal to $c-p$; and C P^2 will be equal to $ap+(c-p)^2$, being the local value of the square of C P, and which is to be a minimum. To accomplish this, it is necessary to find the general value of this function. Taking then the general point P′, and as O I′ equal $p+x$, and I′P′, are its coordinates, then will C P^2 be equal to I′ $P'^2+(c-(p+x))^2$, which is equal to $ap+(c-p)^2+(a-2(c-p))x+x^2$, and this is the general value of the square of C P; and taking x inverse, we have the two general values,

$$\text{C } P^2+(a-2(c-p))x+x^2$$

$$\text{C } P^2-(a-2(c-p))x+x^2. \quad \ldots m.$$

Thus far I have been preparing the question for solution. I have now in the same series, the local and general values of the square of C P. This square itself is the local value, and the whole series is the general value; and the question is, to find what value of p expressed in terms of the parameter a, will make the first term less than the whole of either of these two series, for all mathematical values of the independent variable x, not greater than the indeterminate constant p, or for all values soever of x in the present question, that is, by an equation into which x shall not enter.

Now as the sign that is essential to each term, having as factor any even power of x, will remain the same, whether we take x itself either plus, or minus, while the sign that is essential to each term, having as factor any odd power of x, will change with the change of sign of x; then it is clearly evident, that C P^2 in each series is less than C P^2+x^2, for all mathema-

tical values of x, since the sign of x^2 will always be plus, whether x itself be plus or minus. Therefore, it is demonstrated, that whatever equation or equations, will reduce each general series to the form $C P^2 + x^2$ will resolve the proposed problem. Now, manifestly the equation is simply to make the whole series without the second term equal to the whole series, and in that case we have,

$$C P^2 + x^2 = C P^2 + (a - 2(c-p))x + x^2 \quad \ldots k.$$

$$C P^2 + x^2 = C P^2 - (a - 2(c-p))x + x^2.$$

From both sides taking away the equals, there remains in each series,

$$o = (a - 2(c-p))x.$$

Now, this equation can be resolved in two ways, either by making x zero, or by making the coefficient of x equal to zero. But x is not to be made zero, since the problem is to be resolved independently of any value of x not greater than p; besides, making x zero, proves no more than that x is zero, and that as such, the general series has no existence, by which we are carried again back to the point from which we started. Therefore, since making x zero, cannot lead to any conclusion as to the solution of the question, there remains but to make the coefficient of x equal to zero, and we have in that case,

$$a - 2(c-p) = o, \text{ or } c - p = \frac{a}{2} \ldots n.$$

Replacing $c-p$, by this value in the general series, then since $C P^2$ is equal to $ap + (c-p)^2$, it is reduced to,

$$C P'^2 = C P^2 + x^2 = ap + \frac{a^2}{4} + x^2.$$

which shows that the local function, $ap + \frac{a^2}{4}$, is a minimum, since it is less than the sum of the series by the square of x. And since this function is the square of the normal, it follows that the perpendicular to the tangent, is the shortest line that can be drawn from the same point of the local axis to the parabola; and after this it may readily be demonstrated, that the parabola cannot have two tangent right lines at the same point;

We see that the square of the normal is less than the square of any other line C P′, by the square of x. The maxima and minima are determined in a manner entirely similar in all other functions. If the equation that results from making the coefficient of the first power of the independent variable equal to zero, which I call the limitting equation, is impossible, the function has neither maximum nor minimum.

It is evident, that when the limitting equation is of a quadratic or higher order, that the local function may have several maxima and minima according to the number of the roots of the limitting equation. Therefore, in general, the general function may become equal to the greatest maximum, or least minimum, after which it may increase or decrease without bounds, but according to the theorem of inequality, not for any value of the independent variable, not greater than the local basis.

When there are more than one indeterminate constant, then the coefficient of each independent variable separately is made equal to zero.

Section III.

ON THE RADII OF CURVATURE.

Since the normal is the shortest line from C to the curve, it follows, that a circle described from C as centre, and C P as radius, will no where intersect the curve, while every circle described with a radius BP greater than the normal, will touch the curve at P and intersect it in two other points, except one, which will intersect it at P and nowhere touch it; and this is what is erroneously called the osculating circle, and its radius, the radius of curvature. Calling b the distance B B′, of any point B in the normal line, from the local axis, then will the equation,

$$x^2 - \frac{2ab}{q}x + \frac{a^2b^2}{q^2} - 4qb = o$$

or the equation,

$$x = \frac{ab}{q} \pm \sqrt{4qb} \qquad \ldots 1$$

determine the points P′ and P′, where a circle having B as centre, and B P as radius, intersects the parabola.

If the right hand member of the latter equation be made zero, then x is zero for same value of b, and the values of b will be given that determine the positions of B, when the circle meets the curve in the local P, and these are zero, and, $\frac{4q^3}{a^2}$ or by resolving the equation, by relation to b, we get,

$$b^{\frac{1}{2}} = \frac{q^{\frac{3}{2}}}{a} \pm \sqrt{\frac{q^3}{a^2} + \frac{q}{a}x} .$$

and making x epual to zero, we obtain the same values for b, as above,

$$b^{\frac{1}{2}} = \frac{q^{\frac{3}{2}}}{a} - \frac{q^{\frac{3}{2}}}{a} = o$$

$$b^{\frac{1}{2}} = \frac{q^{\frac{3}{2}}}{a} + \frac{q^{\frac{3}{2}}}{a} = \frac{2q^{\frac{3}{2}}}{a},$$

and therefore, the positions of B are determined by making b, equal to zero, and equal to $\frac{4q^3}{a^2}$. Making b, zero gives the normal for the radius of curvature at P; and making b equal to $\frac{4q^3}{a^2}$, gives the cube of the normal divided by the square of the semi-parameter, for the radius of curvature at the same point. Thus then, since the parabola cannot have two different curvatures at the same point, it is demonstrated, that all those notions of the curvature of curve lines being measured by radii of curvature, are mere fiction.

I shall demonstrate the same thing otherwise. Taking the secant parabola, n P n', and comparing its radius of curvature to that of the given curve, we shall find them to be identically the same.

For in the triangles P C I, P g' g, the cube of the normal PC, divided by the square of the semi-parameter IC, is equal to the cube of the normal P g' divided by the square of the semi-parameter g g, each being equal to P R; and therefore the radii of curvature of the two parabolas are identically the same, and then according to astronomers, their curvatures are the same at the point P. Now the second secant parabola v P v', coincides

at P more nearly with the given parabola, than the first one, since it passes between the first and the given one, both at the convex and at the concave side, and therefore its curvature and its radius of curvature ought to coincide more nearly, and so they do, according to Lagrange and all other astronomers, but then again, according to all astronomers, that is impossible, since there cannot be greater coincidence of curvature than identity; because the two parabolas have the same radius of curvature.

But again, according to astronomers, and according to Lagrange, in the application of his theory of functions to Geometry, the parabola whose equation is of a higher order, coincides more nearly with the given one, than one whose equation is lower, and has its curvature more nearly on an equality with it. Such is the doctrine laid down by all astronomers, but then again, according to astronomers, the first secant parabola, has its curvature the same as that of the given one, since both have a common radius of curvature. So that here are the astronomers against themselves; in fact their doctrine is, and it is the ground work of their philosophy, that of two unequal quantities, the greatest is the least, and the least is the greatest, and so forth.

And again, if in reality, the osculating circle cuts the given curve in the local point, which I shall discuss presently, then in that case, the same curve has different points of equal curvature, since this circle must intersect the given curve in more points than one. Now to determine what takes place in this respect in the parabola, I replace for b, in the equation for x, its value, $\frac{4q^3}{a^2}$ and then it will be,

$$x=\frac{4q^2}{a}\pm\sqrt{\frac{16q^4}{a^2}}=\frac{4q^2}{a}\pm\frac{4q^2}{a}=4p\pm 4p \quad \ldots 2.$$

This gives for x, the value zero, and the value $8p$. Therefore the osculating circle cuts the curve at P, above the local axis, and below this axis, at a point determined by an abscissa equal to nine times p. Therefore if intersections mean equal degrees

of curvature, then every curve has the same curvature at various points, if not like the circle all through it. But I have proved enough to show that the whole theory in this respect, is a mere fiction of astronomers.

It is necessary to remark, that there was nothing contrary to my rules of reasoning, in making x, zero in the equation 1, and restoring it again in the equation 2. For actually in the equation 1, x has two values, and one of these is zero, as is demonstrated in the equation 2, and the other a finite one; and corresponding to these, the indeterminate b has the value zero, and a finite value. But if in any equation the variable has only one value, then after having been once made zero, it cannot be restored again.

All those circumstances about osculating circles, may be investigated in the same manner in all other curves; and in every other curve of the second degree, the radius of curvature of the first secant parabola, is equal to that of the given curve, whether it be a circle, an ellipse, or a hyperbola.

Taking, $\sqrt{r^2-p^2}$, as the local equation of the circle, and developing this, we get the local series,

$$q=r-\frac{p^2}{2r}-\frac{p^4}{8r^3}-\frac{p^6}{16r^5}-\&c.$$

Now, the first two terms of this series is the equation of a parabola tangent to the circle at its origin, and according to astronomers, it must have the same curvature at that point, as the circle, since both have there the same radius of curvature; but then the first three terms form the ordinate of a second parabola, a tangent also to the circle at the origin, and passing to the right and left between the circle and the former parabola, and evidently its curvature is more nearly equal to that of the circle. Therefore, according to the Geometry of the Institute of France, and the Royal Society of London, the greater, is equal to and less than the least. The first four terms form the equation of a third tangent parabola, passing still closer to the circle, and so on without bounds. Taking the local equation of the ellipse to be $\frac{c}{r}\sqrt{r^2-p^2}$, and developing it, we get the local series,

$$c-c\frac{p^2}{2r^2}-c\frac{p^4}{8r^4}-c\frac{p^6}{16r^6}-\&c.$$

By this series, the ellipse, its osculating circle and first tangent parabola, have the same radius of curvature, but each succeeding parabola falls nearer the ellipse than that which next precedes it, and so on. At any point of the ellipse, the first parabola in the general series, has the same radius of curvature

as the ellipse, and to determine the radii of curvature at any point in the ellipse, we have the equations,

$$b = o, b = {}^2\frac{q^3(r^2 - c^2)}{r^2a^4 + q^2c^4}$$

The first gives the normal, as the osculating, radius, which in reality is such, since a circle described with it will touch the curve, and no where intersect it.

The other value of b, gives the pretended radius of curvature of the differentialists.

And now I will propose a question to my friends Poisson and Cauchy of the French Institute, and to my friends the Committee of the Royal Society of London. Will you inform me what is become of the differential calculus now? Where are the fluxtions? And where is the calculus of variations, the admirable and the refined calculus of variations? Where is Taylor's theorem? What is become gentlemen of all these? Will you gentlemen be pleased to inform me in what domecile are the lemmas of Euclid and Newton to be found, and zero upon zero? If you are at a loss gentlemen, I will inform you. I have nailed all that base coin to the counter. Is that worth your attention?

When my new and subtle Geometry comes into the hands of the Geometricians of England and France, how sadly they will mangle it, for they have neither perception, judgment, nor reasoning sufficient to enter into its spirit, and follow it in train.

I will quote now from Lagrange, the method invented by Fermat to resolve the problems of maxima and minima, and who according to Lagrange, was the inventor of the differential calculus. He took the equation $ap - p^2$, which he called his specific equation, but which I call the local or descriptive equation, to determine its maximum.

"On peut regarder Fermat comme le premier inventeur des nouveaux calculs,—l'expression dont on cherche le maximum sera, $ap - p^2$. Ajoutant la quantite arbitraire x, a l'inconnue p, on aura, cette uouvelle expression,

$$a(p+x) - (p+x)^2$$

Egalant ces deux expressions, on a l'equation,

$$ap - p^2 = ap - p^2 + (a - 2p)x - x^2, \ldots 1.$$

Effaçant de part et d'antre les termes communs, $ap - p^2$, et divisant les autres par x, on a,

$$a - 2p - x = o \ldots\ldots 2.$$

ou il faut maintenant supposer, x, nul, ce qui reduit l'equation a,

$$a - 2p = o, \; p = \frac{a}{2}.$$

Il est facile de voir, que la regle dednite du calcul differentiel, qui consiste a egaler a zero la differentielle de l'expression qu'on veut rendre un maximum on un minimum, donne le meme, resultat, parceque le fond est le meme, et que les termes qu'on neglige cemme infiniment petits dans le calcul differentiel, sont ceux qu'on doit supprimer comme nuls dans la methode de Fermat. Sa methode des tangentes depend du meme principe."

Upon this invention I will remark and demonstrate, that the whole reasoning is erroneous. The equations 1 and 2 of Fermat, are totally unconnected with the solution of the question he had to resolve. This question was to divide a given right line a, into two such parts, that their rectangle may be a maximum; or which is the same, to find when the ordinate of a circle, O P A *fig.* 2, of which the diameter O A is equal to a, is the greatest possible. Putting p for O I, then the square of I P is equal to $(a-p)p$, which is the same as $ap-p^2$, and which was to be a maximum, and this was the specific function of Fermat, which is the same as my local function. To determine its maximum, which latter term signifies, to find what part of a is p, when the rectangle $ap-p^2$, is greater than if p had been any other part of a; to determine this, Fermat supposed p, to be increased by an indefinitely small quantity x, and then in place of p, he put $p+x$, and he had $a(p+x)-(p+x)^2$. To this latter function, Fermat gave no name; but if we take from x every idea of its being an infinetesimal, differential, or fluxional quantity, and consider it only in its true geometrical light, as an independent variable, the latter function is then the same as my general function. Fermat, after this, made the local and general functions equal to each other, which produced his equation 1.

$$ap-p^2=ap-p^2+(a-2p)x-x^2,$$

and taking away the equals from both sides, the result was his equation 2.

$$o=a-2p-x.$$

From this he rejected x as nothing, as being infinitely small, and Fermat had then the equation,

$$o=a-2p, \text{ or } p=\frac{a}{2},$$

from which it resulted, that when the given line was bissected, the rectangle was a maximum.

The fallacy of this reasoning is very readily detected. First then as to his equation 1, its only use is to find the point N′ where the partial X′, fig. 2, meets the circle again. Now, since at that point, the local and general ordinates are equal, as I have shewn in the equation, m, in the theory of tangents, we have in that case the equation,

1. $$ap-p^2=ap-p^2+(a-2p)x-x^2$$

which is exactly the equation, 1, of Fermat, with only this difference, that with Fermat, and with the differentialists in general, the quantity x is an infinitesimal, that is, a quantity infinitely smaller than any quantity that could be an object of thought, that is to say, that x is a quantity infinitely smaller than this quantity x itself, since it is an object of thought, while with me, x is in the present case an unknown quantity whose value in terms of p, I propose to determine. Now then to find the value of x in terms of p by this equation, I take away the equals from both sides as Fermat did, and I divide the remainder by x as Fermat did, and the result, with the exception I have stated, is, as Fermat had

$$o=a-2p-x=o$$

being the equation 2 of Fermat. From this I get x to be equal, not to zero, according to Fermat, but to $a-2p$; and since x is equal to P N', then taking P N' equal to $a-2p$, determines the point N' where the partial axis X' intersects the circle again. And now it is quite evident that this result has no connexion whatever with the problem of Fermat.

But, admitting for a moment that $a-2p$ is equal to zero, and that p is equal to half of a, as it really is at the maximum, then the above equation becomes,

$$o=-x.$$

And now, I ask, Poisson and Cauchy of the Institute of France, and the Committee of Science of the Royal Society of London, is x zero in this equation? Certainly not. It is not, because zero is at one side, and x at the other, that x is zero, such an equation is only the sign of impossibility; and in fact this is an impossible equation, shewing, that in the circle, when the local ordinate is a maximum, the partial axis, which in that case is a tangent to the circle, does not meet the circle again. An equation of the form,

$$o=\pm x,$$

how great or small soever the quantity x may be, is the sign of impossibility in simple equations, as the square root of an inverse quantity is in quadratic equations. If the equation 2, was of a quadratic form, and that for some given value of p, the partial axis did not again meet the curve, then the value of x would be imaginary; but this circumstance would show, not that x was zero, which would be absurd, but that the proposed equation was impossible. Besides this, making x zero, would prove no more than that x was made zero, and as such, that the factor $a-2p$, as well as the partial series in which it was, had no existence at all in the calculation, by which we would be carried back to the local equation, from which alone nothing

could be asserted as to the solution of the proposed problem. In fact in all general series, the independent variables increase without bounds to the right and to the left from their origin P. So that Fermat and Poisson and Cauchy of the French Institue, and the Committee of Science of the London Society, are mathematically absurd in making x zero, when the ordinate is at its maximum. And again, they are mathematically absurd in assuming, that, because x is zero, then p is equal to half a. Now I have demonstrated, that making x zero, establishes no relation whatever between a and p. How foolishly absurd then are Poisson and Cauchy of the Institute, and the Committee of Science of the Royal Society, in assuming that because x is zero, then p is half a. But then by this second absurdity, they fall actually on the limitting equation $p=\frac{1}{2}a$, which according to equation n of the theory of maxima and minima, resolves the problem.

Fermat, and many others long before him, knew well the rule for resolving such problems before hand, but he made use of a false science in his attempt at demonstrating that rule. I will say it with all truth, that with regard to the reasoning employed in the method of exhaustions, the theory of fluctions, and the differential and integral calculus, I did not think it possible that man could be led to so much perversion of all the powers of his mind. The most despotic tyrant of barbarous nations, had never kept his slaves in more prostrate servility, than has the method of exhausted quantities of the ancient Greeks kept the geometricians of Europe for more than two thousand years. Nor would I now insult my beautiful geometry, by alluding at all to such foolish modes of thinking, in its discussion, but that they have taken such a prescriptive hold of the human mind, that it requires a bold and determined logic, and a full exposure of their falsehood to expel them from exact science

Now, since x is considered by the writers on the differential calculus, to be smaller than any quantity that could be an object of thought, and as such, is ejected out of the calculation, how can they know the difference between the maximum or minimum and any other state of the function; no differentialist, or fluxionist has ever alluded to that; and in fact it would be impossible to ascertain that difference without the total destruction of the principles of the differential calculus, and for this simple reason, that that difference is expressed in terms of x, which it would be impossible to do if x, were an infinitely small quantity. In the question of Fermat, $\frac{a^2}{4}$ is the

maximum, and $\frac{a^2}{4} - x^2$ is the general state of the function; so that the maximum state exceeds any other, by the square of x, the intermediate part. Here then we have x as large as life, which the differentialists would feign exhaust to death. Is this reasoning worth the attention of the French Academy?

Poisson and Cauchy of France, and the Committee of Science of London, assert, that the fluxion of a function is zero at its maximum and minimum. To which I answer, truly it is zero; and so is it zero in every other state of the function. Geometry has for object the measurement of space; and space has neither fluxion nor differential. Space is a simple idea, one and inseparable, and infinite in extent and duration. Space has no motion nor can it be exhausted into a differential; and what to do I would ask? Would it be to gratify the notions of Euclid and Fermat, Newton and Leibnitz, to make false reasoning, and false philosophy put on the garb of truth? Geometrical magnitudes never change, it is the symbols that change with regard to the magnitudes which they represent. Is this reasoning worth the attention of the Royal Society of London?

Fermat was very near inventing a true science. Instead of making his specific function equal to his infinitesimal series, if he had made that series without its second term equal to the whole series, he would have made use of a true equation as to the solution of his question, and he would have invented a true science, instead of a false one. But, if Fermat had handed down to his followers truth, instead of falsehood, why then, the world would never have seen two great books, the Principia of Newton and the Mecanique Celeste of Laplace, which owe their origin to, and are based upon the false principles of reasoning made use of by Fermat.

In the chapter on fallacies in the treatise on logic by Arch Bishop Whately, he truly observes, that if only one erroneous proposition could be slipt into any process of reasoning, then any conclusion whatever may be arrived at, and the more true propositions were employed, the more likely would the false one be to escape detection. In this way exactly, Newton slipt the lemma of Euclid into his Principia, and then of course that one single false proposition being universally granted as true, even up to the moment, during which I write with one single solitary exception, Newton could then arrive at any conclusion true or false that suited his purpose. As for example, it is not true, that the versed sines of the small arcs, described by the earth in equal times in its motion round the sun, are approximately in the inverse ratio of the squares of the distances of the earth from the sun; for actually those versed sines are

approximately in the inverse ratio of the cubes of those distances. Even on philosophical considerations, it would appear more reasonable to me at least, that, the principal force whatever it is that influences the motions of the planets, should follow a solid ratio rather than a superficial one. But the lemma is the talisman, the philosophers' stone of modern philosophers, by which they can prove, that zero is all things, and that all things are zero, that truth is all falsehood, and that all falsehood is truth. Now according to observation, and independently of all hypotheses, the radius vector of the earth, that is, the line joining the centres of the earth and sun, sweeps over sectors in equal times, the angles in which are inversely as the squares of this radius. Setting out from this data, we may with ten to one less of error than in Newton's reasoning, suppose two sectors described each in a second of time, to be two sectors of unequal circles; and then, it is a Geometrical theorem, that first the arcs of those sectors, are inversely as the radius vector, secondly, that the areas of those sectors are equal to each other, which is Kepler's law, and thirdly, that the versed sines of those sectors, are to each other inversely as the cubes of this radius. The law of Kepler results immediately from this theorem and the observed fact, that the angular velocity of the earth all through its orbit, varies inversely as the square of the radius vector. And now, if as Newton supposes the versed sine in each sector, be the effect due to the central force in a second of time, then it is geometrically demonstrated, that the intensity of the central force, varies inversely as the cube of the distance of the earth from the sun. And now, M. M. Poisson and Cauchy of the Institute of France, and Gentlemen of the Royal Society of London, what have you to object to that law of central force? I have stated a Geometrical theorem, which is in harmony not only with observed facts, but also with the laws of Kepler, and the result contradicts the whole reasoning in the Principia of Newton. I challenge you to controvert that result, but this is certain, that I will not allow you to use the Lemma of Euclid gentlemen; you shall not employ falsehood in reasoning against me, against truth. You will say, no doubt, as the ignorant have often said to myself personally, that the theory of Newton accounts for the phenomena. To which I reply, that it is not true that it does. Euclid placed his lemma at the head of his twelfth book, and that book has not been demonstrated up to the present moment, although with the one exception of myself alone, all the world for more than two thousand years, have considered that book to be as clearly demonstrated as any other book in his Elements; and in the same manner, Newton placed the same lemma at the head of

his Principia, and the consequence is, that there is not one proposition in his Physical Theory of Astronomy, demonstrated to be true. The whole of his theory is only mere assertion without proof; and yet all the world believe otherwise, as they think the reasoning of Newton to be of the same character, as that in the twelfth book of Euclid. And I say so it is; for both are mathematically false, there is not one word of truth in either. Is this reasoning worth the attention of the Royal Society of London and the Institute of France? I assure these Societies, that I hold their judgments on my theory in supreme contempt, for they have been at all times too deeply schooled in false mathematical philosophy to be competent judges of the rules of exact science. I do not censure these Newtonites for being unable to demonstrate their rules, for that could not have been done without the aid of the general and partial forms of functions, of which as not invented until recently they could not have had any idea; but now that these new principles are known and applied; I condemn them for the tenacity with which they still hold fast to falsehood, and not only that, but they are doing so like Gallileo's jury in the teeth of demonstrated truth.

I will now close this section with the discussion of a remarkable curve, N *m* O P A, *fig.* 3, of which the local and general equations are,

$$r^2 \times \mathrm{I\,P} = r^2 q = 3r^2 p - p^3.$$

$$\mathrm{I'\,P'} = y' = q - \frac{3(r^2-p^2)}{r^2}\,x - \frac{3p}{r^2}x^2 + \frac{x^3}{r^2}.$$

If r be the radius of a circle, then the major axis C V is equal to its diameter, and O C to the radius r; and the square of the minor axis O A is equal to three times the square of the radius. If any ordinate I' P' is the chord of an arc of the circle, then the corresponding abscissa O I' is the chord of the third part of that arc; so that this curve trisects any arc from zero to the entire circumference, and may very appropriately be called the Trisector.

The general equation of its tangent is,

$$q - \frac{3(r^2-p^2)}{r^2}\,x$$

To determine the point, m, where this tangent meets the curve again, I make these two terms equal to all four, which gives for x the value $3p$.

To determine the maxium of the ordinate, or more mathematically, to determine what ordinate is the maximum, I make the whole series without the second term equal to the whole series, and in that case it will be,

$$q\frac{3p}{r^2}x^2+\frac{x^3}{r^2}=q-\frac{3(r^2-p^2)}{r^2}x-\frac{3p}{r^2}x^2+\frac{x_3}{r^2}.$$

From this I get at once $p=\pm r$.

Replacing p in the local and general ordinates by r, its value direct, and taking x both plus and minus, the local ordinate then is $2r$, and the two general ordinates for x, plus and minus, will be,

$$2r-\frac{3x^2}{r}-\frac{x_3}{r^2},$$

$$2r-\frac{3x^2}{r}+\frac{x^3}{r^2},$$

In the first of these, the local ordinate $2r$ is a maximum, since it is greater than the whole series for all values of x, by the sum of the remaining two terms. And the local ordinate $2r$, is also a maximum in the second, and greater than the general ordinate, so long as the second term is greater than the third, but after that, the general ordinate is greater than the local one, and continues to increase without bounds. By substituting for p its value inverse, $-r$, we should then have for x, plus and minus, the two general series,

$$-2r+\frac{3x^2}{r}-\frac{x^3}{r^2}$$

$$-2r+\frac{3x^2}{r}+\frac{x^3}{r^2}$$

In these the local ordinate is a minimum. In the second the local ordinate is always a minimum, but in the first it is only so, until the third term becomes equal to the second term, after which the general series decreases without bounds. We see that, as $2r$ is the maximum ordinate, that this ordinate passes through the vertex V of the curve. Let us determine now whether the partial axis will, in this case, intersect the curve again. Equalling the local and general ordinates to each other, we have in that case,

$$2r=2r-\frac{3x^2}{r}-\frac{x^3}{r^2}, \text{ or } x=-3r$$

$$2r=2r-\frac{3x^2}{r}+\frac{x^3}{r^2}, \text{ or } x=3r$$

These values of x show, that the partial axis, when it passes through the vertex, is a tangent to the curve, and that it does not in that position meet the curve again to the right of the vertex, but that it meets it to the left at a point R, determined by a partial abscissa x equal to three times r. And here in this curve the ordinate is a maximum, and yet x is not zero. To determine generally, the points where the partial axis me[illegible]

this curve again, I make the local ordinate equal to the general one, and in that case I have the equation,

$$q = q + \frac{3(r^2-p^2)}{r^2}x - \frac{3px^2}{r^2} - \frac{x^3}{r^2}$$

which gives,

$$o = \frac{3(r^2-p^2)}{r^2} - \frac{3px}{r^2} - \frac{x^2}{r^2}.$$

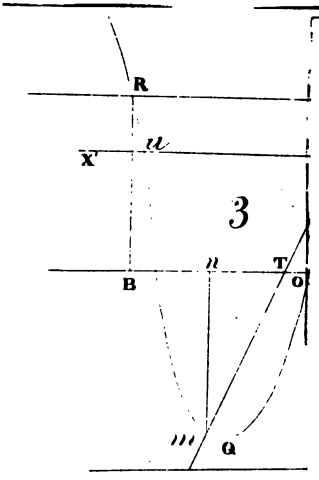
This equation is similar to the equation 1 of Fermat, and obtained in the same way exactly, but then it has no more relation to the maximum ordinate than it has to any other ordidate of the curve; it is a quadratic equation, whose two roots or values of x, determine the two points u and u' where the partial axis meets the curve again, and these roots are,

$$x = \tfrac{3}{2}p \pm \tfrac{1}{2}\sqrt{12r^2-3p^2}.$$

When $2p$ is greater than r, this equation is imaginary, which shows in that case, that the partial axis does not meet the curve but at the local point, to which circumstance I have alluded in a preceeding part of this section. The abscissas O' P, O'u, and O'u', are the three roots of the local or trisecting equation corresponding to the point P.

In concluding this Memoir, I will remark, that the quadrature of curve surfaces, and the cubature of solids as related to three rectangular coordinates, remain as yet untouched by any theory, except that of Partial Functions alone. For the attempt in these cases by the Integral Calculus, is the most absurd and scandalous shuffling that could be made use of. In attempting to find the integral of $zdxdy$, the lemmatists consider r' as given by relation to the sphere, which is absurd; and then they find the definite integral by relation to x alone, which is again absurd; because r' is not a given radius; and they then pretend that $\frac{1}{4}\pi r'^2 dy$, is the differential of the segment of the sphere, cut off by the horizontal plane and the vertical planes of the centre and local point; but it is not true that it is. In fact $\frac{1}{4}\pi r'^2 dy$ is the differential of the volume of a sphere as related to two rectangular coordinates; and thus the Newtonites, adroitly shift from a function of three rectangular coordinates to one of only two, in which they knew the conclusion before hand; than which shifting and shuffling, nothing can be more barefacedly false and absurd. Now, that is the very way that Newton shifts and shuffles in his Principia. He knew what the conclusions ought to be before hand, and then by his talisman the lemma, he managed his reasoning so as to square with those conclusions.

In the second part of my work, the Geometric Base, which I am preparing for publication; I will explain the Theory of Partial Functions in all its generality.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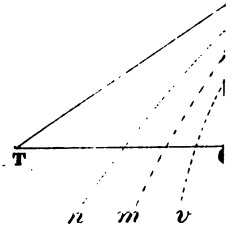
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